

# 4. THE UNCOUNTABLE

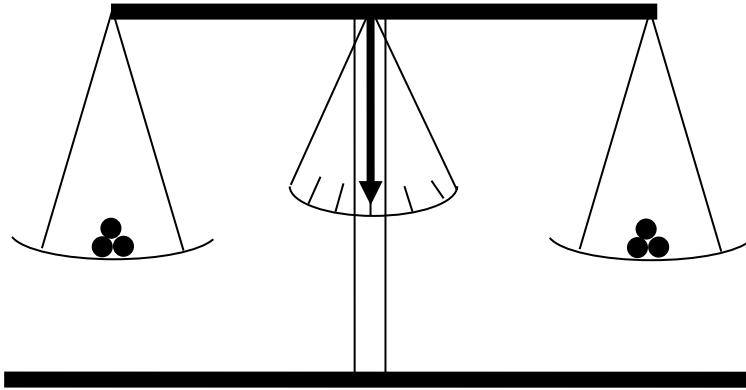
## §4.1. The Same-Number Balance

One might think that counting is the most fundamental concept in all of mathematics. Yet, as we have seen, it is a complex idea built on the even more fundamental one of same-number-as. The one-to-one pairing that defines same-number-as can play a similar role as the old-fashioned beam balance.

This was a device that can only compare the weights of two objects. By itself it can't weigh things absolutely. It merely shows you whether or not the weights are equal. Pairing off in a one-to-one correspondence is the balance we use for counting.

**Two sets have the same-number-as each other if it is possible to pair the elements of one exactly with the elements of the other exactly.**

The reason for the hyphens in 'same-number-as' is because it's a single concept, like 'balance'. As yet we haven't given an independent meaning to the word 'number'. Once we do, we will be able to identify 'same-number-as' with 'same number as' in the sense of each set having a number and those numbers being equal.



## §4.2. Standard Sets

A beam balance can be used for weighing things absolutely, as distinct from comparing weights, only if we have a set of standard weights.



We need some 1 gram weights and 5 gram weights, and so on, perhaps up to 1 kilogram weights. We put combinations of these into one pan of the scales until they balance exactly with the unknown

weight. This enables us to associate a number with the object that we call its ‘weight’.

Before we can count, that is, associate a number with a set to represent its size, we need some standard sets to use in the comparisons. In kindergarten we were introduced to a system of symbols 1, 2, 3, ... and

associated words. These ‘objects’ were initially meaningless things that had a defined ordering. ‘Two’ comes after ‘one’ and then comes ‘three’ and so on. We learnt to recite this list “one two three” as we would a nursery rhyme.

What we were setting up in our brains was a nested collection of standard sets (each fitting inside the other for convenience by just stopping at different places). These standard sets are:

<b>STANDARD SET</b>	<b>SIZE</b>
{ } (empty set)	0
{1}	1
{1, 2}	2
{1, 2, 3}	3
{1, 2, 3, 4}	4
.....	..

What we are doing when we count a set is to select a standard set which pairs off exactly with it. The **size** of the set is just the number associated with it. (For finite sets it’s the last symbol in the list but when we come to infinite standard sets we’ll need to invent new symbols.)

Perhaps as adults we learnt to count in sophisticated ways, grouping things together for convenience. But if we go back to the primitive act of kindergarten counting we point to each object in turn and

call out the next number in the sequence. The last number we reach will automatically be the answer to the counting.

It's important we get it quite clear what the act of counting really means before we introduce our first infinite number.

**The size of a set is the size of the standard set with which it can be put into a one-to-one correspondence**

How many elements are in the set  $\{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$ ? To answer this we find a standard set with which we can put it into one-to-one correspondence. This is what we do as we count. As a young child we might have actually pointed to each of those symbols and counted aloud:

1, 2, 3, 4.

So  $\{1, 2, 3, 4\}$  is the standard set that matches the four suits. The number that we associate with this set is 4, so that is the size of  $\{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$ .

Now with finite standard sets, the associated number is its largest member. But we can't do that with infinite standard sets because they don't have a largest.

### §4.3. The Smallest Infinite Number $\aleph_0$

Are you ready for your first infinite number? We need a standard set and then a symbol to represent its size. What better standard set than the set of all finite numbers  $\{1, 2, 3, \dots\}$ ?

Now for a symbol. You see, we can't use the last element in the list because there isn't one. We could have used the standard 'infinity symbol',  $\infty$ , but that would suggest that this is the only infinite number we're going to get.

Besides it's not the symbol used by Georg Cantor who first investigated infinite counting around the end of the nineteenth century. He chose the first letter of the Hebrew alphabet,  $\aleph$ , and because it was the smallest infinite number he added the subscript '0'. So our list of standard sets has been extended to the following:

STANDARD SET	SIZE
{ } (empty set)	0
{1}	1
{1, 2}	2
{1, 2, 3}	3
.....	..
{1, 2, 3, 4, 5, 6, .....}	$\aleph_0$

## §4.4. In Search of a Bigger Infinite Number (Adding)

Now we begin our long journey, in search of an infinite number bigger than  $\aleph_0$ . With finite numbers we were always able to get a bigger number by adding one.

“My dad's played footy a trillion, trillion times!”

“My dad's played it trillion, trillion plus one times!”

Let's see if  $\aleph_0 + 1$  is a bigger number than  $\aleph_0$ . Well it's certainly not smaller. But could it be just as big? Before we can answer that we must say what we mean to add one to a number, in a way that makes sense for infinite numbers.

When we were learning how to add such finite numbers as 2 and 3 we possibly had a picture of two ducks and three rabbits. Count the ducks. Two. Count the rabbits. Three. How many animals altogether? Before we learnt to add we would have had to count the entire menagerie. One, two, three, four, five. The whole collection of animals matched exactly with our standard set  $\{1, 2, 3, 4, 5\}$  and so its size is 5. We've demonstrated that  $2 + 3 = 5$ .

As time went on we learnt ways of adding without counting. But if pressed for what it means for 37 plus 63 to equal 100 we would have to say something like: "if you take 37 of one type of thing and combine it with 63 of something else we get 100 things altogether".

Addition corresponds to combining two sets of things together. But it's important that the two sets have nothing in common, otherwise we're double counting. So here's our definition of the sum of two numbers.

**To add the numbers  $m$  and  $n$ :**

- (1) Take a set of size  $m$ .**
- (2) Take a set of size  $n$ .**
- (3) Ensure that these sets are disjoint (have no common elements).**
- (4) Combine them into one set (take the union of these disjoint sets).**
- (5) Put this union into 1-1 correspondence with a standard set.**
- (6)  $m + n$  is the number associated with this standard set.**

Let's use this to calculate  $\aleph_0 + 1$ . First we take a set of size  $\aleph_0$ . The standard set  $\{1, 2, 3, \dots\}$  will do. Now a set of size 1. The standard set of size 1 is  $\{1\}$ , but these two sets have '1' in common. So let's change the second one to  $\{0\}$ .

The union of these two sets is  $\{0, 1, 2, 3, \dots\}$ .

Now this certainly appears to be bigger than the set  $\{1, 2, 3, \dots\}$  but is it? No. We can match  $\{0, 1, 2, 3, \dots\}$  off exactly with  $\{1, 2, 3, \dots\}$ . Just write out these sets in rows and each number in the top row pairs off exactly with the one below it:

0	1	2	3	4	5	...
↕	↕	↕	↕	↕	↕	
1	2	3	4	5	6	...

Since neither set has a last element, there is nothing in one row without a mate in the other. According to our definition, therefore, these two sets have the same number of elements. In other words  $\aleph_0 + 1 = \aleph_0$ .

“But that’s absurd. If you add something extra of course you make it bigger!” Careful, you’re revealing your parochialism. It’s just like someone who’s lived all his life in some small outback country town. “Of course, if you go into a bank they’ll know your name!”

You’re no longer in the finite backwoods you’ve been in all your life. This is the big city of the infinite. Some facts you’ve accepted as having universal application, you now find are just curiosities that only work for finite numbers. Other things you’ve learnt *do* extend to the infinite. What can you trust in this strange new world? Just the definitions and logic.

So, contrary to naive intuition, you don’t make an infinite number bigger by adding one to it. Our search for a number bigger than  $\aleph_0$  has so far failed. What about  $\aleph_0 + \aleph_0$ ?

For this we need two disjoint sets of size  $\aleph_0$ . The standard set  $\{1, 2, 3, \dots\}$  will do for one of them and we can take the negative numbers for the other:  $\{-1, -2, -3, \dots\}$ . We can set these out in a table with two infinite rows:

1	2	3	4	5	...
-1	-2	-3	-4	-5	...

Surely these can't be paired off with our standard set for  $\aleph_0$ . To do that we'd have to squeeze both infinite lists into a single one. But that's not difficult. Simply take from each row alternately:

$$1, -1, 2, -2, 3, -3, \dots$$

Nothing is left out, but now that they're in a single infinite list we can pair them off with our standard set  $\{1, 2, 3, \dots\}$ .

1	-1	2	-2	3	-3	...
↕	↕	↕	↕	↕	↕	
1	2	3	4	5	6	...

Note that any infinite set which can be listed in a single list has size  $\aleph_0$ . We just pair the first thing in the list with 1, the second with 2, and so on. Another word that's used for this is **countable**. A set is **countable** if its elements can be listed. Countable sets include the finite ones, as well as these sets which can be put in an infinite list. Our goal is to find an **uncountable** set, whose size will therefore be bigger than  $\aleph_0$ . So far we've failed.

## §4.5. In Search of a Bigger Infinite Number (Multiplication)

We've not yet been successful in finding a number bigger than  $\aleph_0$ . But we were only using addition up till now – a much more powerful operation is multiplication. Perhaps we'll find that  $\aleph_0 \times \aleph_0$  is bigger than  $\aleph_0$ .

What do we mean by multiplication? Repeated addition? But that won't work with infinite numbers for it would mean that  $\aleph_0 \times \aleph_0$  is  $\aleph_0 + \aleph_0 + \dots$  with infinitely many terms. Instead we use the idea of ordered pairs.

A table with 5 rows and 7 columns has 35 cells.


Each cell corresponds to a pair  $(r, c)$  where  $r$  is the number of the row and  $c$  is the number of the column in which it lies.

It's an ordered pair, that is, for example,  $(3, 5) \neq (5, 3)$  because they refer to different cells. So here's the basis for a recipe for multiplying infinite numbers.

**To multiply two numbers  $m$  and  $n$**

- (1) Take a set of size  $m$ .**
- (2) Take a set of size  $n$ .**
- (3) Form the set of all ordered pairs, with the first item in the pair coming from the first set and the second coming from the second set.**
- (5) Put this union into 1-1 correspondence with a standard set.**
- (6)  $m \times n$  is the number associated with this standard set.**

Let's use it to find  $2 \times 3$  and see if we get the answer 6. Take a set of size 2, such as the standard set  $\{1, 2\}$ . Now take a set of size 3, such as the standard set  $\{1, 2, 3\}$ . These sets aren't disjoint, but that doesn't matter for multiplication. The ordered-ness of the pairs will keep them apart.

Now take all ordered pairs with the first item in each pair coming from  $\{1, 2\}$  and the second from  $\{1, 2, 3\}$ . Here they are:

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)

We pair these off with the standard set  $\{1, 2, 3, 4, 5, 6\}$ .

(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)
↕	↕	↕	↕	↕	↕
1	2	3	4	5	6

So we've proved that  $2 \times 3 = 6$ , which is just as well! Our extended definition of multiplication agrees with the way we've always multiplied numbers *but* it gives us a way of multiplying *infinite* numbers.

Now before we tackle  $\aleph_0 \times \aleph_0$ , let's first try  $2 \times \aleph_0$ .

First take a set of size 2.

The standard set  $\{1, 2\}$  will do but for a change we'll take  $\{+, -\}$ .

Take a set of size  $\aleph_0$ .

The standard set  $\{1, 2, 3, \dots\}$  will do.

The pairs  $(x, y)$  where  $x$  is a "+" or a "-" and  $y$  is in  $\{1, 2, 3, \dots\}$  can be put in a table as follows:

(+, 1)	(+, 2)	(+, 3)	...
(-, 1)	(-, 2)	(-, 3)	...

Obviously this is very little different to what we had above and so

$$2 \times \aleph_0 = \aleph_0 + \aleph_0 = \aleph_0$$

as we would expect. So we haven't yet broken the  $\aleph_0$  barrier. But we still have  $\aleph_0 \times \aleph_0$  up our sleeve!

Take two sets of size  $\aleph_0$ . Since they don't have to be disjoint we may as well take the standard set  $\{1, 2, 3, \dots\}$

for both. Now form all ordered pairs. These can be set out in a two-way infinite table:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	...
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	...
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	...
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	...
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	...
...	...	...	...	...	...

Can we squeeze this into a single infinite list? All we have to do is list them by going along the diagonals, starting in the top left-hand corner:

First comes (1, 1), then (2, 1) and (1, 2). Now across to (1, 3) and down the next diagonal and so on.

As a single list this two-way infinite table can be written as a single row:

(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1), (4, 1), .....

$\aleph_0 \times \aleph_0$  elements, all written in a single infinite list, means that  $\aleph_0 \times \aleph_0 = \aleph_0$ . We still haven't succeeded in finding a number bigger than  $\aleph_0$ . Notice, by the way, that fractions can be represented by pairs of whole numbers so the above diagonal process would give us a way of listing all fractions. So while there appear to be

more rational numbers than whole numbers, in fact the set of rational numbers has size  $\aleph_0$ .

## §4.6. The Search for a Bigger Infinite Number (Powers)

If we can't find a number bigger than  $\aleph_0$  we've made a lot of fuss for nothing. But in fact we're just about to reach our goal. Raising numbers to powers is much more powerful an operation than either addition or multiplication. For example  $10 + 10 = 20$ ,  $10 \times 10 = 100$ , but  $10^{10} = 10000000000$ .

You might like to try  $\aleph_0^{\aleph_0}$ , but instead we'll settle for  $2^{\aleph_0}$ , which is easier to discuss and is just as big. How can we give a meaning to  $2^n$  for a number  $n$ . Multiplying 2 by itself  $n$  times is satisfactory for finite  $n$  but not if  $n$  is infinite. The secret to the correct definition lies in the concept of subsets.

One set is a **subset** of another if everything in the first set is an element, or member, of the second set. For example the set of all women in the world is a subset of the set of all people.

We allow a set to be a subset of itself, so the set of all people is another subset of the set of all people. We even include the empty set as a subset. The set of all

people who are over 1000 years old is a subset of the set of all people. It's just that it happens to be empty.

Take a set with two elements, say  $\{1, 2\}$ . How many subsets does it have? Well, what are the subsets of  $\{1, 2\}$ ? First there's the empty set  $\{\}$ . then the subsets  $\{1\}$ , and  $\{2\}$ , and finally the set itself  $\{1, 2\}$ . There are 4 subsets. This will be true of any set with 2 elements.

Take a set with 3 elements such as  $\{1, 2, 3\}$ . What are the subsets? They are:

$\{\}$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$  and finally  $\{1, 2, 3\}$  itself. How many? Eight.

So a set with 2 elements has 4 subsets, a set with 3 elements has 8. Is there a pattern? Yes, a set with  $n$  elements has  $2^n$  different subsets, at least if  $n$  is finite.

A quick way to see this is to consider that each subset corresponds to a decision for each element as to whether or not it's in the subset. Imagine a sergeant lining up his men and asking for volunteers for latrine duty. In true military fashion it's the sergeant who does the volunteering. As he goes along the row of men he says, "you're in, not you, nor you, yes I want you, no, no, no, yes, ...". There are  $n$  choices, each a choice from two alternatives, so altogether there are  $2^n$  possible subsets. Now this, which is a fact for finite numbers, can be taken as the definition of  $2^n$  for infinite numbers.

**To raise 2 to the power  $n$**

- (1) Take a set of size  $n$ .**
- (2) Form the set of all its subsets.**
- (3) Put this set of all subsets into 1-1 correspondence with a standard set.**
- (4)  $2^n$  is the number associated with this standard set.**

## **§4.7. The Number $2^{\aleph_0}$ is bigger than $\aleph_0$**

Powers of 2 grow quickly and it is a simple fact that  $2^n$  is bigger than  $n$ , for finite  $n$ . But we'll show, by a cunning argument, that  $2^n$  is bigger than  $n$  for *all* numbers,  $n$ , finite or infinite.

Showing that  $2^n$  is bigger than  $n$  involves two steps. We'll first prove that  $2^{\aleph_0}$  is at least as big as  $\aleph_0$ .

$$2^{\aleph_0} \geq \aleph_0$$

“At least as big as” means finding a way of pairing off all the elements of a set with *some* of its subsets. That's easy – you just pair off each element in the set with the corresponding set with one element. The elements of  $\{1, 2, 3\}$  can be paired off with some of its subsets, namely  $1 \leftrightarrow \{1\}$ ,  $2 \leftrightarrow \{2\}$ ,  $3 \leftrightarrow \{3\}$ . The fact that there are subsets left over, such as  $\{1, 2\}$  etc, shows that  $2^n$  is bigger than  $n$ , for finite  $n$ , but, as we have seen, having things left over after a pairing doesn't necessarily mean

‘bigger’ because there could be another pairing that leaves nothing over.

$$2^{\aleph_0} \neq \aleph_0$$

The proof that  $2^{\aleph_0}$  and  $\aleph_0$  are different runs along very familiar lines. We suppose that  $2^{\aleph_0} = \aleph_0$  and get a contradiction.

So suppose then that  $2^{\aleph_0} = \aleph_0$ .

Let  $\mathbb{N}$  be the set  $\{1, 2, 3, \dots\}$ . This has size  $\aleph_0$ . To say that  $2^{\aleph_0} = \aleph_0$  means that there must be an exact pairing off of the elements of  $\mathbb{N}$  with the subsets. Every element has a corresponding subset and vice versa.

For a given number  $n$ , one of two things will be true. Either  $n$  belongs to the subset that it corresponds to, or it does not.

For example one of the subsets of  $\mathbb{N}$  will be  $\mathbb{N}$  itself, and of course the corresponding element must belong to it. At the other end of the scale, one of the subsets is the empty set and the corresponding element can't belong to it.

If, for example, 3 corresponds to the set  $\{1, 4, 5\}$  then 3 will not belong to the set that it corresponds to. If 7 corresponds to the set of all odd numbers then 7 will belong to the set it corresponds to.

Suppose we call those elements which belong to the subset they correspond to, **internal** elements. Those which lie outside their corresponding subset will be called **external** elements.

In symbols, if we denote the subset that corresponds to the element  $x$  by  $S(x)$ , and use the symbol ' $\in$ ' to denote 'is a member of' and ' $\notin$ ' to denote 'is not a member of', then we can describe these properties of being internal and external as follows:

$$\begin{aligned}x \text{ is } \textit{internal} & \text{ if } x \in S(x) \\x \text{ is } \textit{external} & \text{ if } x \notin S(x)\end{aligned}$$

Of course whether an element is internal or external would depend on the particular one-to-one correspondence. But if somebody claimed to have a way of pairing off all the elements of a set with all of its subsets (rash claim!) it's perfectly reasonable to expect that they could tell us whether any given element is internal or external.

Suppose, for argument sake, that somebody claimed to have paired off all the elements of  $\{1, 2, 3, \dots\}$  with all of its subsets. Then, in principle, they must have a list such as the following:

1	↔	{11, 32, 117}
2	↔	set of powers of 2
3	↔	empty set
4	↔	set of all multiples of 3
5	↔	set of prime numbers
...	...	.....
3427	↔	set of all numbers
...	...	.....
185367	↔	set of all even numbers
...	...	.....
673867	↔	set of all external numbers
9		
...	...	.....

If this was indeed such a list then 1, 3, 4 and 185367 would be external. They would lie outside their corresponding set. The elements 2, 5, 3427 would be internal. If somebody claimed to have such a list it would also be reasonable, in principle, for us to ask where the set of all external elements appears in the list – what number does this set correspond to? There is such a subset and so if the pairing is exact, as claimed, there’s a corresponding element. In the above example we are supposing that it’s 6738679.

Is 6738679 itself an internal number or an external one? It has to be one or the other.

**If it's internal** then it belongs to the set that it corresponds to, that is, it belongs to the set of all external numbers which would make it external. That's nonsense. So it can't be internal.

But wait! **If it is external**, it's a member of the set of external numbers. So it *does* belong to the set it corresponds to. But this would make it internal! That's nonsense too!

If it is internal then it is external. If it is external, it is internal. One big resounding contradiction! And that contradiction all rests on the assumption that we started with, that the elements could be paired off with the subsets. Therefore they can't be. That is, the number of elements of any set cannot be paired off exactly with the subsets. The same argument can be used for any set.

**$2^n > n$  for all counting numbers, finite or infinite.**

## **§4.8. The Universe of Infinite Numbers**

So  $2^{\aleph_0}$  is bigger than  $\aleph_0$ . We call it  $\aleph_1$ . Actually  $\aleph_1$  is usually defined to mean the next infinite number after  $\aleph_0$ . But nobody knows whether that's  $2^{\aleph_0}$  or not. So it seems reasonable to define  $\aleph_1$  to be  $2^{\aleph_0}$ .

But if we do that, what if somebody finds an infinite number between  $\aleph_0$  and  $2^{\aleph_0}$ ? We'd then have to call it  $\aleph_{1/2}$  or something like that. Relax! That will never happen. Nobody will ever find any numbers between  $\aleph_0$

and  $2^{\aleph_0}$ . How can we be so sure? Because it has been *proved* that the existence of something between the two is unprovable. Surely that means there aren't any! Not exactly, because nobody has been able to prove that the next number after  $\aleph_0$  is indeed  $2^{\aleph_0}$ . What's more nobody ever will, because a proof exists that shows that it is impossible to prove the next number after  $\aleph_0$  is  $2^{\aleph_0}$  and there's a proof that it's impossible to show that it isn't.

Amazing stuff, but all quite logical. It has been proved that the question of whether there's a counting number between  $\aleph_0$  and  $2^{\aleph_0}$  is **undecidable**. So if nobody will ever be able to find one (that's different to saying that there aren't any) it makes sense to assume that there aren't any (not the same as proving that there aren't any).

The statement that nothing exists between  $\aleph_0$  and  $2^{\aleph_0}$  is called the **Continuum Hypothesis**. It's a hypothesis, not a fact. But it isn't a conjecture that will be settled one day. It will forever remain an hypothesis. You could say that whether it is true or not is a matter of faith.

"I believe in the Continuum Hypothesis," your creed might run. Fine. That's perfectly consistent with everything else we know about mathematics. But the opposite view is equally logical. I suppose the proper stance to take would be that of an agnostic.

On the other hand, even though it can never be proved, there's a metalogical argument in favour of believing in the Continuum Hypothesis. Since nobody will ever find an actual example of a number between the two (for if they did the matter would be decidable, contradicting the proof of the matter's undecidability) then for all practical purposes there *isn't* one. Though this falls short of an actual proof of non-existence, it seems a reasonable position to take.

So taking  $\aleph_1$  to be  $2^{\aleph_0}$  we can then use the same argument as above to show that  $2^{\aleph_1}$  is bigger than  $\aleph_1$  and so on. That means there is a whole infinity of infinite numbers:

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \dots$$

each bigger than the one before.

If we set out to construct a catalogue of counting numbers we'd start with two rows in our table:

0	1	2	3	4	...
$\aleph_0$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	...

But, as they say in the TV advertisements for steaks knives, "there's more!" If you take a whole collection of sets, one for each of the infinite numbers in the second row of this table, and combine all these sets into one huge set (we call this "taking the union") the size of that set will be at least as big as any number in the row, and hence

must actually be *bigger* than anything in the row (think about it!). This will then give us a number bigger than anything in the two rows, so we can use it to start a third row.

But then by taking successive powers of 2 we can work our way along the third row to produce a third infinite sequences of infinite numbers. But wait, there's more. In the same way we got from the second row to the third we can get from the third to a fourth row, and a fifth and so on.

So our catalogue of numbers, all but the first row being infinite, now covers an entire infinite page, with infinitely many infinite rows. But there's still more. There exists a number bigger than any number on the page and so we can start a second page, and a third, and so on until our catalogue occupies infinitely many pages, each with infinitely many infinite rows.

But why stop at one such volume. We can have infinitely many volumes on an infinitely long shelf, and infinitely many such shelves .... The human mind is a wonderful thing to be able to conceive, and even think logically about, such expansive concepts.

Is there any practical use to all this? Such a question brings us back to earth with a thud, even though the answer is "yes". Mathematicians have a real use for

knowing about  $\aleph_0$ ,  $\aleph_1$  and to some extent about  $\aleph_2$ . We could live without the others. The number  $\aleph_1$  is the number of points on a line, or the number of real numbers. The number  $\aleph_2$  is the number of functions from the set of real numbers to itself.

Where does  $\aleph_0^{\aleph_0}$  fit into all this? Is it bigger than  $2^{\aleph_0}$ ? No, in fact it can be shown that it's just the same as  $2^{\aleph_0}$ .

# INTERLUDE: RADIO FEATURE

## “Beyond the Finite”

**MUSIC:** *Also Sprach Zarathustra* by Richard Strauss

**MALE VOICE:** Beyond the familiar numbers 1, 2, 3, ... of the kindergarten, beyond the hundreds of the cricket scoreboard, beyond the millions, tens of millions, millions of millions of economic statistics, beyond the billions of billions, billions of billions of billions of astronomy ..... beyond all finite numbers, lies ..... the infinite!

Man, imprisoned though he is in a finite world, is able to glimpse the infinity beyond, through the tiny barred windows of religion, philosophy and mathematics.

**FEMALE VOICE:** Infinity is an ideal that one can approach but never reach.

**PRESENTER:** This popular view of the infinite has grown out of the mathematical concept of infinite limits which underpins the calculus, but it's not the only insight that mathematics can give us into the nature of the infinite. A somewhat more recent development, though known to mathematicians for over a hundred years, has yet to make its imprint on the popular mind.

In the 1890s, Georg Cantor extended the concept of counting to infinite collections and came up with a theory of transfinite numbers. Not just a single unattainable

infinity, but a whole infinity of bigger and bigger infinite numbers.

To appreciate this mind-boggling concept I want you to come with me on a journey – a journey of the imagination – a journey beyond the finite to the infinite world of Infinland.

**MUSIC:** *Enigma Variations* by Edward Elgar

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Once upon a time there was, in a far-off place, a kingdom called Infinland in which there lived a race of creatures, rather similar to men only much smaller, called Infins. There was nothing very remarkable about these Infins except that there were infinitely many of them. I don't mean that their population was exploding at an ever-increasing rate, approaching infinity. It was infinite and always had been.

Infins were happy little creatures. And so they should be for their infinite land was ideally suited to cope with an infinite population. There were none of the annoying shortages that we experience in our overcrowded world.

Take housing for example. Every Infin had his own house. When a pair of young Infins got married and left home, they were allowed to choose any house in the kingdom. Never mind that it was already occupied. The family living there, and all those beyond that point on that

side of the infinitely long street were obliged to move up to the next house to make room. Because the street was infinitely long there was no last house in the street to be pushed off the end. A marvellous system! You and I might resent the frequent moving, but the Infins had never known any other way.



They lived on both sides of a single road – the East-West Road which stretched for ever in either direction. To the north and south of this eternal road were the royal gardens belonging to King Aleph II. These, too, stretched for ever to the north and to the south. Situated right across the East-West Road at one point was the castle of King Aleph, separating it into an eastern section and a western section. So that the inhabitants of the eastern section could have contact with those in the western section, the king very kindly provided a right of way across the castle grounds – for which he charged a modest toll.

Now the Infins had not always lived on the East-West Road. In



fact several years before, the Infins' houses had completely covered what were now King Aleph's fields. Then, the castle in the middle enclosed a modest garden within its walls. Around this ran the road called "The Circle". From this stretched the East-West Road and the North-South Road – both going on forever in either direction. These were the major highways of the kingdom, but the whole area was crossed by a network of minor roads as well. Some ran north and south and others east and west. All roads stretched on forever in both directions.

One day, King Aleph decided that his modest garden was not big enough. He decreed, therefore, that henceforth Infins must only live on the northern side of the East-West Road.

In any ordinary kingdom this would have created a severe housing problem. However, as the Infin kingdom was infinite it was possible to rehouse everybody on the East-West Road. On the day appointed, the King sent a messenger around the kingdom. Starting with the houses nearest the castle, he travelled in a spiral fashion around the castle, moving further out all the time. Calling at each house (including those already on the East-West Road) he gave out their new addresses in order. Travelling in this spiral fashion, the messenger was able to call upon every house. In an ordinary kingdom this would have taken him forever. In fact he would have never got to the end. But Infins can move infinitely quickly if they have a mind to and so the job was done in next to no time at all. And

because there were infinitely many houses on the northern side of the East-West Road he never ran out of new addresses to give the families.

The traffic chaos was unimaginable as families moved to their new houses. Even families already living on this highway were unhappy about the change for they had to move much further along the road.

All the houses left unoccupied were demolished and the land became gardens for the king's private use. Since that time traffic on the East-West Road has been in a permanent state of chaos and Infins who had once been close neighbours now lived vast distances apart.

This mean and despotic act was just one of the many carried out by King Aleph II. As you may have gathered, the King was very unpopular. Yet he was allowed to rule as the people respected the ancient charter laid down by the much-loved grandfather of King Aleph II, Aleph Zero. In this charter it was laid down that his descendants would be entitled to rule so long as they carried out their duties as Lord of Committees.

That's another peculiar thing about the Infins – their love of committees. They liked nothing better than forming committees. They formed committees at work and committees at



school. Every Infin family was organised into committees and subcommittees and sub-sub committees.

Infin waiting at bus-stops would immediately elect a chairman and ask for the minutes of the last meeting to be read out.

Another curious feature of the Infin committees is that the identity of the committee depended solely on the collection of Infins present. If any Infin was absent from a regular meeting the committee was deemed to be a different one.

This made the call for apologies redundant because, by definition, every member was automatically present. But it did complicate the reading of the minutes because they had to recall when and where and under what circumstances that exact collection of Infins last met.

A committee that met by chance at a bus-stop one day may have had exactly the same membership as the one that happened to be together in the same laundromat many months previously and so constituted the same committee.

The King, as Lord of Committees, had the statutory right and duty to choose a chairman for every committee. However he had to respect the rule, laid down by Aleph Zero:

**No Infin may be chairman to more than one committee.**

So long as he carried out this task the Infins allowed him to continue to rule. But if he ever defaulted he lost the right to rule.

Now Infins are notorious for their poor memories, the King included. So it often happened that he forgot that he had chosen a certain Infin previously and made the mistake of choosing him to chair a totally different committee. The trouble is that although Infins often thought they could remember that someone had doubled up, they couldn't quite remember. And although they always took minutes of their meetings, they were so disorganised that they could never find them later when they needed them.

So the King continued to get away with his ineptitude. He made out that he consulted a large volume in which he had written down all possible committees and his chosen chairmen but the truth was he just chose the first name that came into his head or, if he couldn't remember any name he just pointed to someone and said, "you there, I appoint you".

The chairman could be, and often was, chosen by the King from within the committee. Such a chairman was called an **internal** chairman. At other times the King chose a chairman from outside, known as an **external** chairman.

An external chairman was not actually co-opted because changing the composition would change the committee into a quite different one which would of course require a different chairman. No, the Infins were

not so stupid as to allow themselves to be caught in a recursive trap like that. An external chairman chaired but always from the outside.

Now although he usually made a random choice, King Aleph was occasionally put in the position of having to be very crafty in his choice. Once, in an attempt to overthrow the King, the Infins called together a committee consisting of everybody except the King. The King clearly couldn't suffer the indignity of being the only one excluded and so he chose himself as external chairman of that committee.

Another attempt was made to overthrow the King by Count Able, one of the noblemen of the kingdom. Count Able maintained that King Aleph, by himself, constituted a committee of one and asked the King to select a chairman. Now King Aleph remembered that he had nominated himself as chairman of the Every-One-Except-The-King committee, and just in case Count Able remembered too, he thought it safest to select an external chairman. So he chose Count Able himself as that external chairman. And, for his insolence he cast Count Able into solitary confinement in the dungeons.

As he was being dragged off he screamed for the King to appoint a chairman of the Solitary Confinement Committee that consisted of just Count Able. You see, he couldn't be named as an internal chairman – he was already external chairman to the King-Of-Infinland Committee and he hoped that whoever got to be external chairman might be able to help him escape.

Of course the King merely appointed the soldier who kept guard outside Count Able's cell as external chairman so such hopes of escape came to naught.

Nothing ever lasts for ever, not even in Infinland, and eventually Count Able was released. But during his confinement he had hit upon a cunning plan to trap the King.

He called a meeting of the Every-One-Except-The-King Committee. Of course, as external chairman, the king had to come too. So the whole infinite population of Infinland crowded into the Great Meeting Hall of the castle. Count Able respectfully asked leave to put a question to the King, and leave was granted.

“Oh noble King, Lord of Committees, you have the royal privilege of choosing chairmen for all committees both actual and potential.”



“Indeed I do. I have it all written down in my Book of Chairmen here.”

“And most noble King, you may not choose the same Infin to chair more than one committee.”

“Quite right. That's why I have it all written down.”

“So nobody can possibly be both an external and an internal chairman.”

“Certainly not for that would violate my grandfather’s charter and I would lose the right to rule.”

“So you’d be able to consult your book and tell us who among us you have chosen to be external chairmen.”

“That is so. At the back of the book I have an index that lists the name of every Infin and next to those whom I’ve honoured by choosing them as chairmen I have recorded the letter ‘E’ to denote that they are an external chairman or ‘I’ to denote an internal chairman.”

Of course the King was making all this up. The great book was completely blank but nobody was permitted to look inside. However the more he said the more poor King Aleph was playing into Count Able’s hands.

“Then I wish to call a meeting of the Committee of all External Chairmen. Would your highness please read out the names.”

The King should have insisted that this would take too long but, eager to demonstrate his power as Lord of Committees he foolishly co-operated far too readily.

“Certainly,” he said lifting up the great book and indicating a line of division. “You my people on my right are the external chairmen of Infinland – oh, plus myself and Count Able. All others are dismissed.”

The supposed internal chairmen filed out muttering. Many of them half-remembered having been appointed external chairman of some committee in the past but their memories were not sufficiently strong to contradict the King. Someone else remarked on the

extreme coincidence of the external chairmen happening to be all standing on one side of the Great Hall. But even an extreme coincidence is not a water-tight proof of fraud, not if it's the King who is supposed to be guilty.

“So your Infinite Majesty we here comprise all the external chairmen of your kingdom. You're sure?”

“Of course I'm sure. It's all written down in my book.”

“Each one of us is the external chairman of some committee?”

“That's what I said.”

“And those who've left are either internal chairmen ...”.

“... or they're not chairmen of anything”. The King finished that sentence but he had no idea of the next one!

“Then I ask you, as Lord of Committees to appoint a chairman of this present committee.”

The king pretended to consult his book while he thought this out carefully. He sensed a trap but he knew he was safe because of the incredibly bad memories of Infins. If he couldn't remember whether this present collection of Infins had ever assembled before nor could anyone else. So even if he was inconsistent with what he'd done in the past nobody would be able to remember. No he was quite safe.

He was just about to point to the nearest Infin and announce that he was chairman when the thought struck him. That would make him an internal chairman but no internal chairman remained. He'd confirmed that just a

moment ago. The memory of an Infin is bad, but it's not that bad! They could all remember that only external chairmen remained.

That was close. He'd nearly put his foot in it. But fortunately he was a match for Count Able. All he had to do was remember the name of someone who'd left. He consulted his completely blank book.

"I appoint the last Infin who left as the chairman of this committee."

Count Able had him, and King Aleph knew it as soon as he had said this. The King hid his face behind the book to hide his blushes.

"So if he's not here, that would make him an external chairman of this committee. But all external chairmen are present in this hall. Q.E.D."

Immediately there was an uproar and the Infin Revolt had begun. Very few Infins knew what the letters Q.E.D. stood for but they had been told that those three letters would be a sign that the revolution had begun. Perhaps Q.E.D. meant "Quick, everyone destroy" because they did just that. The castle was destroyed and Count Able was declared the next King.

At his induction he made just two conditions to his accepting the crown.

"I ask two things as your new King. Firstly I wish not to assume the duties of Lord of Committees and secondly I ask that throughout the rest of my life I be granted ownership of any house which is situated next door to one that I already own."

The Infins thought that both of these were very reasonable requests and the Count became King.

When they later discovered that being granted any house that was situated next door to one he already owned resulted in King Able owning the entire eastern half of the East-West Road, they weren't at all put out. Those on the western half simply moved to double their street number and those moving out from the eastern half simply occupied the odd-numbered houses in between.

